

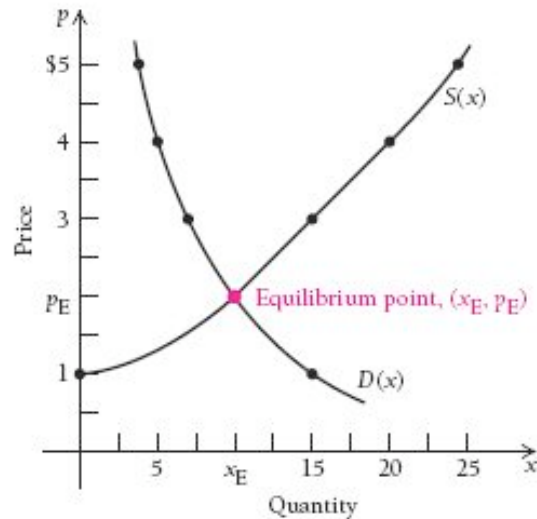
5.1 Economics Application: Consumer Surplus and Producer Surplus.

The **Demand Curve** $p = D(x)$ – *from the Consumer's Perspective.*

Generally, the lower the price of a product, the more the consumer's will demand the product. That is, high prices reduce demand and low prices raise demand. So, generally, $p = D(x)$ is a decreasing function.

The **Supply Curve** $p = S(x)$ – *from the Producer's Perspective.*

Generally, the higher the price of a product, the more the producer's are willing to supply. That is, high prices increase supply and low prices decrease supply. So, generally, the supply curve is an increasing function.



The **Equilibrium Point** (x_E, p_E) is the intersection of the supply and demand curves.

Utility, U , is an economic idea. When a consumer receives x units of a product a certain amount of pleasure, or utility, is derived from it. For example, seeing four movies in a month gives you more utility than seeing one movie.

Example A

The graph shown is the demand curve for the number of movies Sam will see in a month. The total benefit (utility; measured in \$) Sam receives for seeing one movie in a month is the area under the curve over the interval $[0, 1]$.

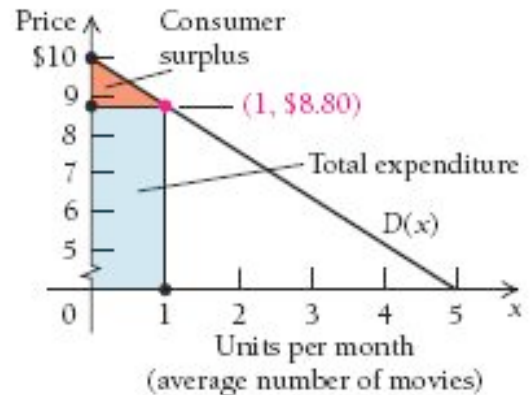


FIGURE 1

- Compute Sam's utility for seeing one movie per month.
- If the average cost of a movie is \$8.80, what is Sam's average expenditure for the month?
- What is the benefit Sam received for seeing the movie, but did not pay for? This is called the **consumer surplus**.

- Compute Sam's utility (benefit) for seeing two movies per month.

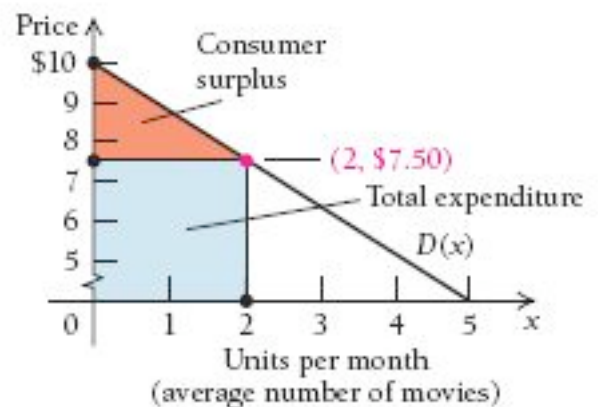


FIGURE 2

- If the average cost of a movie when seeing two movies per month is \$7.50, what is Sam's average expenditure for the month?
- What is the benefit Sam received, but did not pay for? This is called the **consumer surplus**.

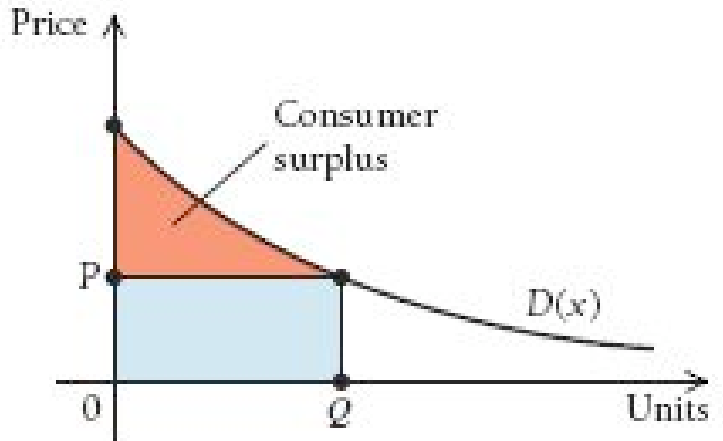
Consumer Surplus

The **consumer surplus** is the total utility minus the total expenditure. In general, suppose that $p = D(x)$ is the demand function for a commodity and $D(Q) = P$.

Then the consumer surplus is defined for the point (Q, P)

as

$$\int_0^Q D(x) dx - P \cdot Q.$$



Example 1 Find the consumer surplus for the demand function given by $D(x) = (x - 5)^2$ when $x = 3$.

Producer Surplus

Now let's look at the supply curve for movies. At a price of \$0 per ticket, the supplier is willing to supply 0 movies per month. At a price of \$5.75 per ticket, the supplier is willing to supply 1 movie per month.

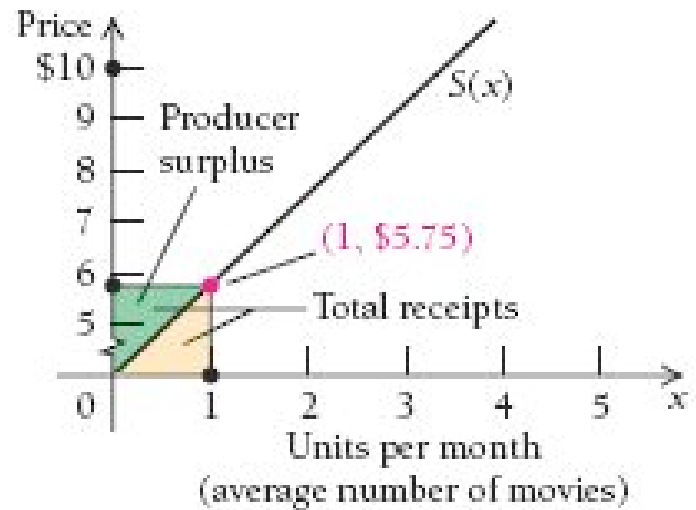


FIGURE 3

- What are the total receipts the theater (producer) receives for supplying Sam with one movie per month?
- What does it cost the theater (producer) to show one movie per month?
- Compute the total receipts minus the total cost? Explain its meaning in this application.
- The area of the green triangle is \$2.875 and is called the *producer surplus*; it represents the [supplier's] surplus over its cost; it is a contribution to profit. The **producer surplus** is the total receipts minus the total cost. It is the benefit the producer receives when supplying more units at a higher price, rather than supplying fewer units at a lower price. Find the producer surplus for supplying Sam with one movie per month for \$5.75?

- e. Now suppose that the theater is willing to supply two movies per month at a ticket price of \$7.50 each. What are the total receipts for supplying Sam with two movies per month?

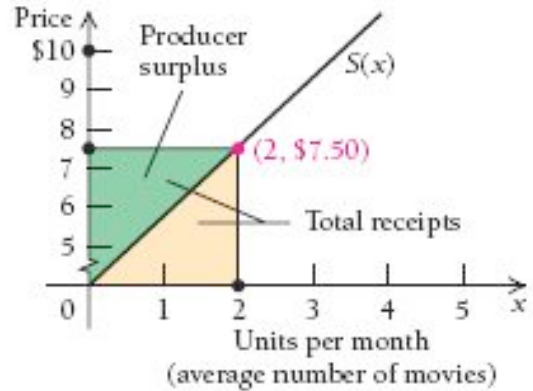


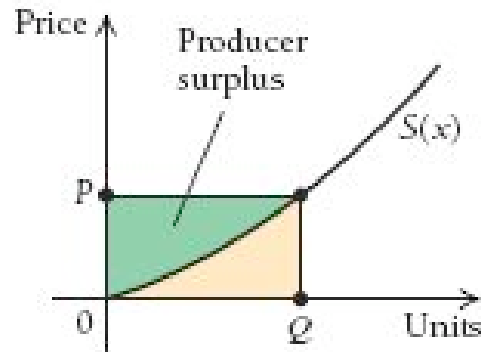
FIGURE 4

- f. What does it cost the theater (producer) to show Sam two movies per month?
- g. Find the producer surplus for supplying Sam with two movies per month?

Producer Surplus

Now suppose the supply function is $p = S(x)$ and the theater shows Sam Q movies when the price is P , that is $S(Q) = P$. The **producer surplus** is the total receipts minus the area under the curve and is given by

$$P \cdot Q - \int_0^Q S(x) dx.$$



Example 2 Find the producer surplus for $S(x) = x^2 + x + 3$ when $x = 3$.

Equilibrium Point

The **equilibrium point** (x_E, p_E) is where the supply and demand curves intersect. This is that point where sellers and buyers come together and purchases and sales are actually made. It is the point where consumer and producer surpluses are defined.

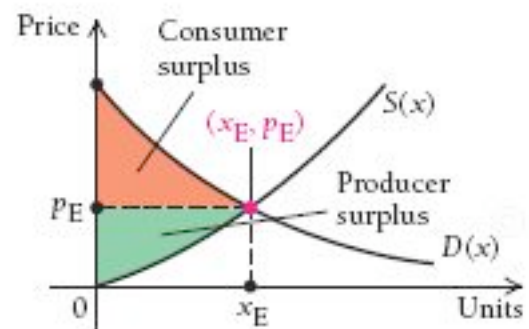
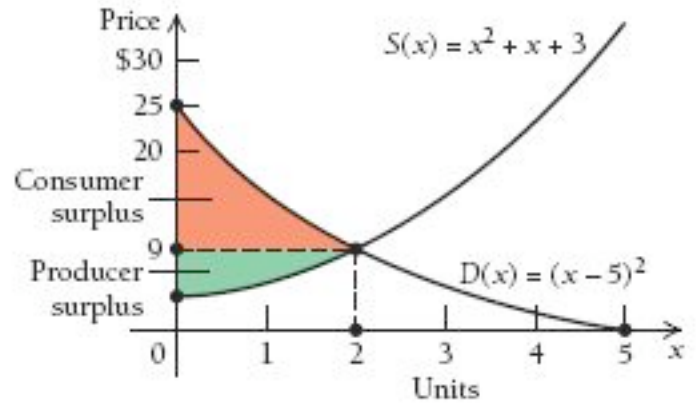


FIGURE 5

Example 3

Given $D(x) = (x - 5)^2$ and $S(x) = x^2 + x + 3$, find each of the following.

a. The equilibrium point.



b. The consumer surplus at the equilibrium point.

c. The producer surplus at the equilibrium point.

5.2 Applications of $\int_0^T P_0 e^{kt} dt$ and $\int_0^T P_0 e^{-kt} dt$

Recall: $\frac{dP}{dt} = kP$ is equivalent to $P(t) = P_0 e^{kt}$

Example 1 *Business: Growth in an Investment.* Find the balance in a savings account 3 years after \$1000 is invested at 8%, compounded continuously.

Future Value of a Continuous Money Flow

If the yearly flow of money into an investment is given by $R(t)$, then the **future value of the continuous money flow** at interest rate k , compounded continuously, over T years is given by

$$\int_0^T R(t) \cdot e^{kt} dt$$

Example 2 *Business: Future Value of a Continuous Money Flow.* Find the future value of a continuous money flow if \$1000 per year flows at a constant rate into an account paying 8%, compounded annually, for 15 years.

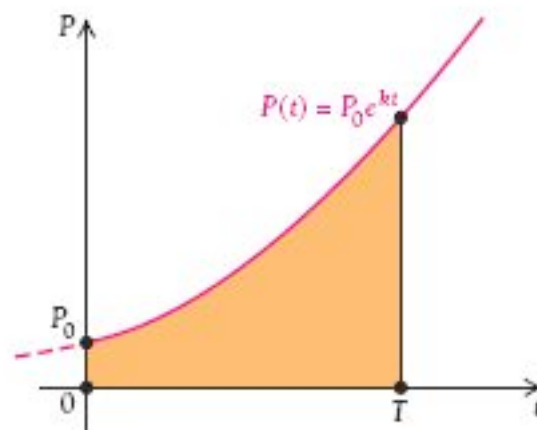
Example 3 *Business: Continuous Money Flow.* Consider the continuous money flow into an investment at the constant rate of P_0 dollars per year. What should P_0 be so that the amount of a continuous flow over 20 years at an interest rate of 8%, compounded continuously, will be \$10,000?

Consumption of a Natural Resource

Suppose that $P(t) = P_0 e^{kt}$ is the annual consumption of the resource in year t . Since the consumption grows exponentially at a growth rate of k , the total consumption of the resource after T years is given by

$$\int_0^T P_0 e^{kt} dt = \frac{P_0}{k} (e^{kT} - 1),$$

where $P(0) = P_0$.



Example 4 *Physical Science: Gold Mining.* In 2000 ($t = 0$) world gold production was 2547 metric tons, and it was growing exponentially at a rate of 0.6% per year. If the growth continues at this rate, how many tons of gold will be produced from 2000 to 2013?

Example 5 *Physical Science: Depletion of World's Gold Reserves.* In 2000 ($t = 0$) world gold reserves were estimated to be 77,000 metric tons. Assuming growth rate of production given in example 4 continues and no new reserves are discovered, when will the world reserves of gold be depleted?

Present Value

The **present value**, P_0 , of an amount P , when P_0 is invested at interest rate k , compounded continuously, and due t years later, is given by

$$P_0 = Pe^{-kt}$$

Example 5 *Business: Present Value.* Find the present value of \$200,000 due 25 years from now, at 8.7% compounded continuously.

Accumulated Present Value

The **accumulated present value**, A , of a continuous money flow into an investment at a rate of $R(t)$ dollars per year from now until T years in the future is given by

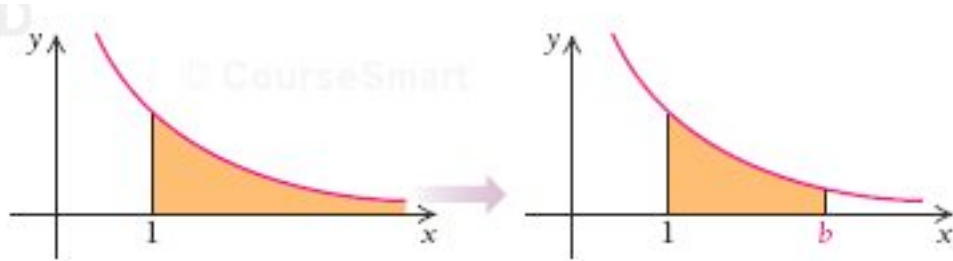
$$A = \int_0^T R(t) e^{-kt} dt,$$

where k is interest rate, compounded continuously.

Example 7 *Business: Accumulated Present Value.* Find the accumulated present value of an investment over a 5-year period if there is a continuous money flow of \$2400 per year and the interest rate is 14% compounded continuously.

5.3 Improper Integrals

Example A *Area Under a Curve*. Find the area of the region under the graph of $y = 1/x^2$ over the interval $[1, \infty]$.



Example B *Area Under a Curve*. Find the area of the region under the graph of $y = 1/x$ over the interval $[1, \infty]$.

Improper Integral

An **improper integral** is one in which at least one of the limits of integration is ∞ or $-\infty$.

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

If the limit exists, the improper integral is **convergent**. Otherwise, the integral is **divergent**.

Example 1 *Evaluate Improper Integrals.* Evaluate each integral or state that it is divergent:

a. $\int_0^{\infty} 4e^{-2x} dx$

b. $\int_1^{\infty} \ln x dx$

Improper Integral

An **improper integral** is one in which at least one of the limits of integration is ∞ or $-\infty$.

$$1. \quad \int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

$$2. \quad \int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$$

$$3. \quad \int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$$

Example C *Evaluate Improper Integrals.* Evaluate each integral or state that it is divergent:

a. $\int_{-\infty}^0 e^{2x} dx$

b. $\int_{-\infty}^{\infty} xe^{-x^2} dx$

Accumulated Present Value

The **accumulated present value**, A , of a continuous money flow into an investment of P dollars per year from now until T years in the future is given by

$$A = \int_0^T P e^{-kt} dt = \frac{P}{k} (1 - e^{-kt}),$$

where k is interest rate, compounded continuously. If the money flow is to continue perpetually (forever), then take the limit as t approaches infinity.

Example D *Business: Accumulated Present Value.*

a. Find the accumulated present value of an investment over a 5-year period if there is a continuous money flow of \$2400 per year and the interest rate is 14% compounded continuously.

b. Find the accumulated present value of an investment with a perpetual money flow of \$2400. Assume the interest rate is 14% compounded continuously.

Accumulated Present Value

The **accumulated present value**, A , of a continuous money flow into an investment of P dollars per year from now until T years in the future is given by

$$A = \int_0^T P e^{-kt} dt = \frac{P}{k} (1 - e^{-kt}),$$

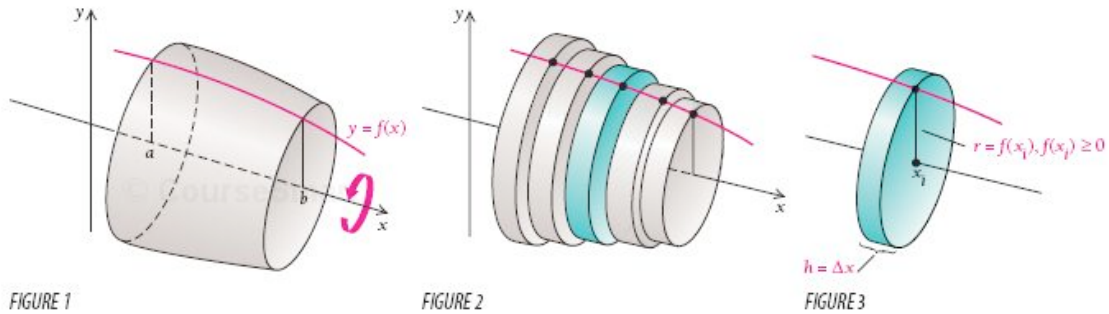
where k is interest rate, compounded continuously. If the money flow is to continue perpetually (forever), then take the limit as t approaches infinity and

$$A = \int_0^{\infty} P e^{-kt} dt = \lim_{t \rightarrow \infty} \left[\frac{P}{k} (1 - e^{-kt}) \right] = \frac{P}{k}$$

$$A = \frac{P}{k}$$

Example 2 *Business: Accumulated Present Value.* Find the accumulated present value of an investment for which there is a perpetual continuous money flow of \$2000 per year. Assume the interest is 8% compounded continuously.

5.6 Volume of a Solid of Revolution

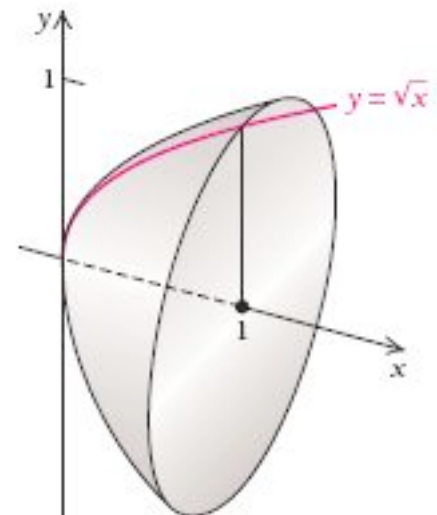


Theorem 1

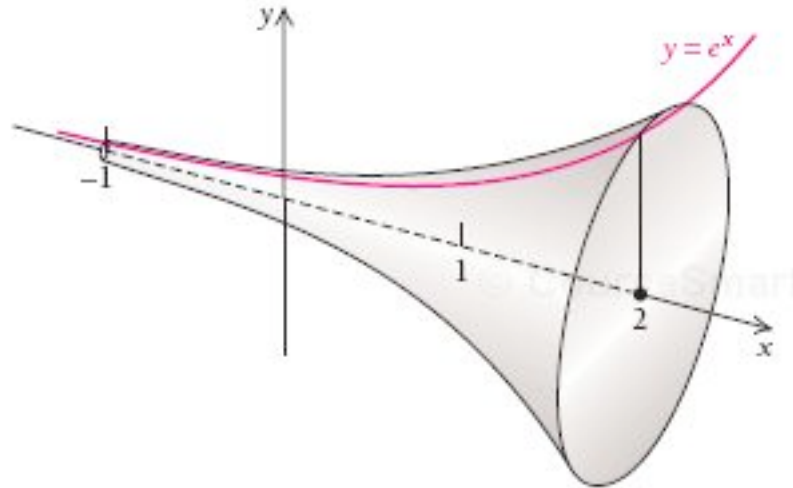
For a continuous function f defined on $[a, b]$, the **volume, V , of the solid of revolution** obtained by rotating about the x -axis the area between the graph of f and $[a, b]$ is given by

$$V = \int_a^b \pi [f(x)]^2 dx$$

Example 1 *Volume of a Solid of Revolution.* Find the volume of the solid of revolution generated by rotating about the x -axis the region under the graph of $y = \sqrt{x}$ from $x = 0$ to $x = 1$.



Example 2 *Volume of a Solid of Revolution.* Find the volume of the solid of revolution generated by rotating about the x -axis the region under the graph of $y = e^x$ from $x = -1$ to $x = 2$.



5.7 Differential Equations

A **differential equation** is an equation that contains a derivative.

Example A

Show that the differential equation, from Chapter 3,

$$\frac{dP}{dt} = kP, \text{ with } P(0) = P_0$$

has the solution $P(t) = P_0 e^{kt}$.

Example 1 Solve $y' = 3x$

Example 2 Solve $f'(x) = e^x + 5x - x^{1/2}$, given that $f(0) = 8$.

Example 3 *Verifying a Solution.* Show that $y = 4e^x + 5e^{3x}$ is a solution of $y'' - 4y' + 3y = 0$.

Example B *Separation of Variables.* Solve $\frac{dy}{dx} = 2xy$.

Example 4 *Separation of Variables.* Solve $3y^2 \frac{dy}{dx} + x = 0$, where $y = 5$ when $x = 0$.

Example 5 *Separation of Variables.* Solve $\frac{dy}{dx} = \frac{x}{y}$

Example 6 *Separation of Variables.* Solve $y' = x - xy$

34. Capital expansion. *Domar's capital expansion model is*

$$\frac{dI}{dt} = h k I,$$

where I is the investment, h is the investment productivity (constant), k is the marginal productivity to the consumer (constant), and t is the time.

- a) Use separation of variables to solve the differential equation.
- b) Rewrite the solution in terms of the condition $I_0 = I(0)$.