7.1.1 Add and Subtract Polynomials

Monomials
A **monomial** is a number, a variable, or a product of numbers and variables.

Examples of Monomials
a. 7 A number

b. $\frac{3}{4}d$ A product of a number and a variable

c. $-12xy^3$ A product of a number and variables

d. $\frac{-x}{12}$ A product of the number $-\frac{1}{12}$ and $x$

e. $\sqrt{11}$ An irrational number

Examples of Non-Monomials
a. $3\sqrt{x}$ Cannot be written as a product of variables.

b. $\frac{2c}{d^2}$ This is a quotient of variables - not a product.

c. $-2x^{\frac{3}{4}}$ and $4y^{-2}$ Exponents on variables must be non-negative integers $\{0, 1, 2, 3, \ldots\}$
Polynomials
A polynomial is a variable expression in which the terms (the addends) are monomials.

i. A one-term polynomial is a monomial.
   e.g. $9x^2$ or $-12$

ii. A two-term polynomial is a binomial.
   e.g. $-14x^2 - 13$ the terms are ____ and ____.

iii. A three-term polynomial is a trinomial.
   e.g. $8x^3 - 3x + 5$ the terms are ____, ____ and ____.

Descending Order
By convention, the terms of a polynomial in one variable are arranged so that the exponents of the variable decrease from left to right with the constant term listed last. This is called descending order.

For example, $3 - x^2 + x$ simplifies to $-x^2 + x + 3$.

Example 1 Rewrite the polynomial in descending order.
$2x - 3x^4 + 5x^2 - 9 =$

Degree of a Polynomial
The degree of a polynomial in one variable is the largest exponent that occurs on any variable.

Example 2 State the degree and type of each polynomial on this page.
Example 3  Add or Subtract. Then simplify.
a.  \((3x^2 - x + 2) + (-3x^3 + 4x^2 - 5)\)

b.  \((2x^2 - x - 7) - (-x^3 + 4x + 13)\)

Example 4
Circle True if the statement is always true, otherwise circle False.

True     False   All terms of a polynomial are monomials.
True     False   The degree of a constant polynomial is one.
True     False   All like terms have the same coefficient and the same variable part.
True     False   Subtraction is addition of the opposite.
True     False   The opposite of \(-4x^2 - 5x\) is \(4x^2 + 5\).

Example 5
Write a trinomial in one variable in descending order that has
  i.   degree 3, and
  ii.  coefficients -2 and 7, and
  iii. no like term
7.2 Multiplication of Monomials

Recall Factors and Exponents
1. **Factors** in an expression are the things that are multiplied together.
   
   e.g. $3x^2y$ has factors ______________________

2. In the **exponential expression** $x^5$, $x$ is the base and 5 is the exponent. In fact, $x^5 = x \cdot x \cdot x \cdot x \cdot x$ can be thought of as “five factors of $x$.”

Example 1
Write the expression in expanded form and simplify:

$x^4 \cdot x^3$

**Rule for Multiplying Exponential Expressions**

If $m$ and $n$ are integers, then

$$x^m \cdot x^n = x^{m+n}$$

e.g. $a^2 \cdot a = a^2 \cdot a^1 = a^{2+1} = a^3$

In words this says “to multiply monomials that have the same base, retain the base and add the exponents.”

Example 2  Simplify

a. $a^2 \cdot a^5$

b. $(2xy)(-4x^2y)$
Example 1

a. Simplify \((-x^2 y)(-4x^2 y^4)\)

b. Simplify \((-3a^2 b^2)(6a^3 b)\)

c. Simplify \((x + 2y)(x + 2y)^2\)

d. Simplify \((x^2 + 2x - 4)^3(x^2 + 2x - 4)^2\)
7.2.2 Simplify Powers of Monomials

Write each expression in expanded form and simplify.

a. \((x^2)^3 = (x^2)(x^2)(x^2) = \) 

b. \((x^4y^3)^2 = (x^4y^3)(x^4y^3) = \)

Notice that multiplying the exponent of each factor inside the parentheses by the exponent outside the parentheses gives the same result.

Rule for Simplifying Powers of Exponential Expressions
If \(m, n\), and \(p\) are integers, then

1. \((x^m)^n = x^{mn}\) e.g. \((a^4)^5 = a^{20}\)

2. \((x^m y^n)^p = x^{mp} y^{np}\) e.g. \((-2xy^2)^3 = -8x^3y^6\)

Example 1
Simplify

a. \((a^4)^3\)  \hspace{2cm} b. \((3x^2y)^3\) 

c. \((-2x)(-4xy^2)^3\)  \hspace{2cm} d. \((-5c)(-2c^3d)^3\)
7.3 Polynomial Multiplication

7.3.1 Multiply: Monomial · Polynomial
7.3.2 Multiply: Binomial · Polynomial
7.3.3 FOIL Method: Binomial · Binomial
7.3.4 Multiply: \((a + b)(a - b), (a + b)^2, (a - b)^2\)
7.3.5 Application Problems

Example 1 Use the distributive property to multiply.

a. \(-2x(x^2 - 4x - 3)\)

b. \((-5x + 4)(-2x)\)

c. \((-2y + 3)(-11y)\)

d. \(-z^2(4z^2 + 2z - 7)\)
7.3.2 Multiply: Binomial \times Polynomial

Example 2 Multiply and simplify.

a. \((y - 2)(y^2 + 3y + 1)\)

b. \((2b^2 - b + 1)(2b + 3)\)

c. \((2y^3 + 3y^2 - 4)(-5y - 1)\)

d. \((3x^3 - 2x^2 + x - 5)(2x + 5)\)

f. \((2x^2 - 3x + 4)^2\)
7.3.3 FOIL Method: Binomial-Binomial

Example 3 Multiply and simplify.

\[(2x + 3)(x + 5) = 2x(x + 5) + 3(x + 5)\]

\[= 2x^2 + 10x + 3x + 15\]

\[= 2x^2 + 13x + 15\]

**F** Multiply the First binomial terms: \(2x(x) = 2x^2\)

**O** Multiply the Outside binomial terms: \(2x(5) = 10x\)

**I** Multiply the Inside binomial terms: \(3(x) = 3x\)

**L** Multiply the Last binomial terms: \(3(5) = 15\)

b. \((4x - 3)(3x - 2)\)

c. \((5y - 4)(y - 7)\)

d. \((2x - 3)(4x + 9)\)

e. \((2a - 4)^2\)
7.3.4 Multiplying the Sum and Difference of Two Terms

Product of the Sum and Difference of Two Terms
The binomial $a + b$ is the sum of two terms and $a - b$ is the
difference of two terms. The product of a sum and
difference of two terms is in the form called a **difference of
two squares**, $a^2 - b^2$. That is,

$$(a + b)(a - b) = a^2 - b^2$$

e.g. $(2x + 5)(2x - 5) = (2x)^2 - (5)^2 = 4x^2 - 25$

<table>
<thead>
<tr>
<th>Sum of Two Terms</th>
<th>Difference of Two Terms</th>
<th>Difference of two squares</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x + y)$</td>
<td>$(x - y)$</td>
<td>$x^2 - y^2$</td>
</tr>
<tr>
<td>$(x + 2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(3a - 4)$</td>
<td></td>
</tr>
<tr>
<td>$(-2y + 5)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Example 4** Multiply and simplify.

a. $(x + 4)(x - 4)$

b. $(2x + 3)(2x - 3)$

c. $(-3x + 8)(-3x - 8)$

d. $(-4a - 3b)(-a + 3b)$
7.3.4b \( (\text{Binomial})^2 = \text{Perfect-Square Trinomial} \)

**Perfect-Square Trinomial**

A **perfect-square integer** is formed by squaring an integer. That is, 16 is a perfect square integer because \( 4^2 = 16 \). Squaring any binomial forms a **perfect-square trinomial**.

\( (\text{Binomial})^2 = \text{Perfect-Square Trinomial} \)

1. \( (a+b)^2 = a^2 + 2ab + b^2 \)
2. \( (a-b)^2 = a^2 - 2ab + b^2 \)

In words, the right side of both equations is read “the first binomial term squared plus twice the product of the two binomial terms plus the second binomial term squared.”

\[
\begin{align*}
(a+b)^2 &= \text{First binomial term squared} + \text{Twice the product of the two binomial terms} + \text{Second binomial term squared} \\
(a-b)^2 &= \text{First binomial term squared} - \text{Twice the product of the two binomial terms} + \text{Second binomial term squared}
\end{align*}
\]

**Example 4** Multiply (i.e. write in expanded form).

a. Multiply \( (2a+b)^2 \)

b. Multiply \( (3x+2y)^2 \)
(Binomial)² = Perfect-Square Trinomial
1. \((a + b)^2 = a^2 + 2ab + b^2\)
2. \((a - b)^2 = a^2 - 2ab + b^2\)

Example 5  Multiply
a. \((5x - 2)^2\)

b. \((6x - y)^2\)

c. \((3a - 2b)^2\)

d. \((3c - 5d)^2\)

Skip the Application Problems in section 7.3.5
### 7.4.1 Division of Monomials

**Example 1**  Write the numerator and denominator in expanded form and simplify.

\[ \frac{x^5}{x^2} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x} = \]

**Rule for Dividing Exponential Expressions**

If \( m \) and \( n \) are integers and \( x \neq 0 \), then

\[ \frac{x^m}{x^n} = x^{m-n} \quad \text{e.g.} \quad \frac{x^5}{x^3} = x^{5-3} = x^2 \]

**Example 2**

a. Simplify using \( \frac{x^m}{x^n} = x^{m-n} \).

\[ \frac{a^5}{a^5} \]

b. Simplify by dividing out all common factors.

\[ \frac{a^5}{a^5} \]

**Zero as an Exponent**

If \( x \neq 0 \), then \( x^0 = 1 \). The expression \( 0^0 \) is undefined. In words this says anything (except zero) with an exponent of zero is 1.

**Example 2A**

a. Simplify \((2xy^2)^0\)

b. Simplify \(-3x^2)^0\)
Example 3

a. Simplify using \( \frac{x^m}{x^n} = x^{m-n} \). \( \frac{x^3}{x^5} \)

b. Simplify by dividing out all common factors. \( \frac{x^3}{x^5} \)

Since \( \frac{x^3}{x^5} = x^{3-5} = x^{-2} \) and \( \frac{x^3}{x^5} = \frac{x \cdot x \cdot x}{x \cdot x \cdot x \cdot x} = \frac{1}{x^2} \), it follows that \( x^{-2} = \frac{1}{x^2} \). Notice that a negative exponent does nothing to the sign of the expression - it only means to take the reciprocal of the expression and change the sign of the exponent.

Negative Exponents
If \( n \) is a positive integer and \( x \neq 0 \), then

\[ x^{-n} = \frac{1}{x^n} \quad \text{and} \quad \frac{1}{x^{-n}} = x^n \]

Example 4
Write each expression with only positive exponent and evaluate.

a. \( 2^{-3} \)  

b. \( \frac{3^{-3}}{3^2} \)  

c. \( \frac{2^{-2}}{2^3} \)  

d. \( \frac{4}{4^{-2}} \)
Exponent Rules – A Summary

1. \(x^m \cdot x^n = x^{m+n}\)

2. \((x^m)^n = x^{m \cdot n}\)

3. \((x^m y^n)^p = x^{m \cdot p} y^{n \cdot p}\)

4. \(\frac{x^m}{x^n} = x^{m-n}\)

5. \(x^0 = 1, x \neq 0\)

6. \(x^{-n} = \frac{1}{x^n}, x \neq 0\)

An exponential expression in **simplest form** has only positive exponents.

**Example 5**  Simplify

a. \(a^{-7} b^3\)

b. \(\frac{x^{-4} y^6}{xy^2}\)

c. \(\frac{b^8}{a^{-5} b^6}\)
d. \[ \frac{-35a^6b^{-2}}{25a^{-2}b^5} \]

e. \((−2x)(3x^{-2})^{-3}\)

f. \[ \frac{12x^{-8}y^4}{−16xy^{-3}} \]

g. \(-5x^{-2}\)

h. \((−5x)^{-2}\)
### Definition of Scientific Notation

A number expressed in **scientific notation** is in the form

\[ a \cdot 10^n \]

Where \( 1 \leq |a| < 10 \), and \( n \) is an integer.

### Example 6

Write each in scientific notation.

<table>
<thead>
<tr>
<th>Decimal Notation</th>
<th>Scientific Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. 240,000</td>
<td></td>
</tr>
<tr>
<td>b. 93,000,000,000</td>
<td></td>
</tr>
<tr>
<td>c. 3,450,000</td>
<td></td>
</tr>
<tr>
<td>c. 844,200,000</td>
<td></td>
</tr>
<tr>
<td>d. 0.00040</td>
<td></td>
</tr>
<tr>
<td>e. 0.00000867</td>
<td></td>
</tr>
<tr>
<td>f. 0.000000000165</td>
<td></td>
</tr>
<tr>
<td>g. 0.25</td>
<td></td>
</tr>
</tbody>
</table>
Example 7  Write each number in decimal form.

<table>
<thead>
<tr>
<th>Scientific Notation</th>
<th>Decimal Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $2.345 \cdot 10^5$</td>
<td></td>
</tr>
<tr>
<td>b. $5.6 \cdot 10^{-8}$</td>
<td></td>
</tr>
<tr>
<td>c. $4 \cdot 10^7$</td>
<td></td>
</tr>
<tr>
<td>d. $3.0425 \cdot 10^{-7}$</td>
<td></td>
</tr>
</tbody>
</table>

Example 8  Simplify

a. $(3 \cdot 10^5)(1.2 \cdot 10^{-7})$  
   b. $(2.4 \cdot 10^{-9})(1.3 \cdot 10^{3})$

c. $\frac{7.2 \cdot 10^{13}}{2.4 \cdot 10^{-3}}$  
   d. $\frac{5.4 \cdot 10^{-2}}{1.8 \cdot 10^{-4}}$
7.5.1 Divide: \[
\text{Polynomial \over Monomial}
\]

Fact \[
\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \quad \text{e.g.} \quad \frac{12+4}{4} = \]

Example 9 Divide

a. \[
\frac{6x^2 + 12x}{2x}
\]

b. \[
\frac{6x^3 - 3x^2 + 9x}{3x}
\]

c. \[
\frac{4x^3y - 8x^2y^2 - 4xy^3}{2xy}
\]

d. \[
\frac{24x^2y^2 + 18xy + 6y}{6xy}
\]
7.5 Division Algorithm for Polynomials

Using the division algorithm we find that \( \frac{19}{3} = 19 \div 3 = 6 \frac{1}{3} \).

\[
\begin{array}{c|ccc}
\text{Divisor} & 3 & \rightarrow & 19 \\
\hline
\text{Dividend} & 6 \\
\text{Quotient} \\
-18 & \\
\text{Remainder} & 1
\end{array}
\]

The division algorithm for polynomials states the same thing:

\[
\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}
\]

\[
\frac{19}{3} = 6 + \frac{1}{3} = 6 \frac{1}{3}
\]

Example 1 Divide \( \frac{6x^3 + x^2 - 18x + 10}{3x - 4} \)
Example 2  Divide $\frac{x^3 - x + 10}{x - 4}$

Example 3  Divide $\frac{x^4 + x^2 - 6}{x + 2}$