9.1.1 Simplify Rational Expressions (Algebraic Fractions)

Definition

A **rational expression** (or **algebraic fraction)** is a fraction with a polynomial in the numerator and and a nonzero polynomial in the denominator.

e.g. $\frac{-4}{x}$, $\frac{2y^3-4}{-5y+9}$, $\frac{-4a^3-7}{a^2-3a}$, $\frac{x-4}{x^2+3x-28}$

Definition

The **domain of a rational expression (algebraic fraction)** is the set of all real numbers except the value(s) of the variable that result in division by zero when substituted into the expression. To find the values of the variable *excluded* from the domain, set the denominator equal to zero and solve.

Example 1

Find the values excluded from domain of each algebraic fraction above.

a. $\frac{-4}{x}$

b. $\frac{2y^3-4}{-5y+9}$

c. $\frac{-4a^3-7}{a^2-3a}$

d. $\frac{x-4}{x^2+3x-28}$

Definition

To **simplify a rational expression (an algebraic fraction)** means to write the fraction so that there are no common factors other than 1 or -1.

Steps to simplify an algebraic fraction.

1. Factor the numerator and denominator.
2. Divide out all common factors.

Example 2 (*Note* Monomials are already in factored form)

1. Simplify $\dfrac{4x^3y^5}{6x^4y^2}$

2. Simplify $\dfrac{6x^5y}{12x^2y^3}$

3. Simplify $\dfrac{x^2-4}{x^2-2x-8}$

4. Simplify $\dfrac{x-4}{4-x}$

Recall that the opposite of $a-b$ is $-(a-b)=-a+b$ which is often written as $b-a$. Furthermore, the quotient of any expression and its opposite is negative one.

i.e. $$\frac{a-b}{b-a}=-1$$

Example 3

1. Simplify $\dfrac{(3-4x)(x-7)}{(x+5)(4x-3)}$

2. Simplify $\dfrac{14-7x}{x^3-2x^2}$

4. Simplify $\dfrac{9a-3}{3a^2+11a-4}$

5. Simplify $\dfrac{x^2+8x+7}{x^2-4x-5}$

6. Simplify $\dfrac{10+3x-x^2}{x^2+6x+8}$

7. Simplify $\dfrac{9-x^2}{x^2+x-12}$

8. Simplify $\dfrac{a^2+2a-24}{16-a^2}$

9.1.2 Multiply Rational Expressions (Algebraic Fractions)

Fraction Multiplication Property $\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$

Steps to multiply algebraic fractions

1. Factor all numerators and denominators.
2. Multiply by applying the fraction multiplication property. That is, multiply the numerators together and the denominators together.
3. Simplify by dividing out all common factors.

It is often more convenient to interchange steps 2 and 3.

Example 1

1. Multiply: $\frac{4a^2b^3}{15x^5y^3}\cdot\frac{25x^4y^2}{16ab}$

2. Multiply: $\frac{12x^4y^5}{7a^2b^4}\cdot\frac{14a^4b^5}{9x^2y^3}$

3. Multiply: $\dfrac{10x^2-15x}{12x-8}\cdot\dfrac{3x-2}{20x-25}$

4. Multiply: $\dfrac{12x^2+3x}{10x-15}\cdot\dfrac{8x-12}{9x+18}$

5. Multiply: $\dfrac{x^2+2x-35}{21-4x-x^2}\cdot\dfrac{x^2+3x-18}{x^2+9x+18}$

6. Multiply: $\dfrac{2x^2+5x+2}{2x^2+7x+3}\cdot\dfrac{x^2-7x-30}{40+6x-x^2}$

7. Multiply: $\dfrac{6x^2-11x+4}{6x^2+x-2}\cdot\dfrac{12x^2+11x+2}{8x^2+14x+3}$

9.1.3 Divide Rational Expressions (Algebraic Fractions)

Definition

The **reciprocal** of a fraction is the fraction created by interchanging the numerator and denominator.

e.g.	Fraction	Reciprocal
	$\frac{x}{y}$	
	c^2	
	$\frac{b-5}{y}$	

Definition of Fraction Division

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$$

Divide and, as always, simplify.

1. $\frac{ab^2 - 4a^2b}{c^3} \div \frac{4b^2 - 16ab}{c^5}$

2. $\frac{x^2}{4yz^2} \div \frac{x}{6yz - 3y^2}$

9.2.1 Express Fractions in terms of their Least Common Denominator (LCD)

Definition

The **least common multiple (LCM)** of two or more numbers is the smallest number that has the given numbers as factors.

e.g. Find the LCM of 12 and 18.

a. the multiples of 18 are

b. the multiples of 12 are

c. the common multiples of 18 and 12 are

d. the least common multiple (LCM) of 18 and 12 is

Definition

The **LCM of two or more polynomials** is the simplest polynomial in which the given polynomials are factors. It is the product of the LCM of the coefficients and every variable factor the greatest number of times is occurs in *any* of the given polynomials.

Steps to find the LCM of Two or More Polynomials

1. Factor each polynomial. Monomials are already in factored form.
2. Find the LCM of the coefficients.
3. Find the product of each variable factor the greatest number of times it occurs in any factored polynomial from step 1.
4. The LCM is the product of the factors from steps 2 and 3.

Examples

1. Find the LCM of $8xy^3$ and $-6x^2y$.

2. Find the LCM of $8b^3$ and $-20ab$.

Find the LCM of

1. $3x^2$
   $4x-12$

2. $2x^2y$
   $5x^2+10x$

3. a^2-6a+9
   $3+2a-a^2$

4. $a^2 - 6a + 9$

   $a^2 - 2a - 3$

5. $m^2 + 3m - 18$

   $3 - m$

   $m + 6$

6. $x^2 - 5x + 6$

   $1 - x$

   $x - 6$

9.2.2 Rewrite Fractions as equivalent fractions in terms of their least common denominator (LCD)

Definition

The **least common denominator (LCD)** of two or more fractions is the LCM of the denominators of the fractions.

Steps to Write Algebraic Factions in Terms of their Least Common Denominator (LCD).

1. Factor each denominator.
2. Find the LCD of the fractions; the LCM of the denominators.
3. Multiply the numerator and denominator of each algebraic fraction by each factor in the LCD that is not already a factor in the denominator of the fraction.

Write each pair of algebraic fractions in terms of their LCD.

1. $\frac{-2}{xy}$

   $\frac{5}{x^2 y}$

2. $\frac{a+3}{4a^2}$

   $\frac{a-1}{8ab}$

3. $\dfrac{x-2}{4x^2y}$

$\dfrac{2x+1}{18xy^2}$

4. $\dfrac{4a+1}{2a-a^2}$

$\dfrac{5}{a^2+a-6}$

5. $\dfrac{c+4}{c^2-3c-10}$

$\dfrac{2c}{25-c^2}$

9.3.1 Add and Subtract Rational Expressions with a common denominator

Fact $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$

To add (or subtract) two fractions that have a common denominator, simply add (or subtract) the numerators and retain the common denominator. Then simplify the resulting sum (or difference).

Add or subtract, then simplify.

1. $\frac{4a}{28} + \frac{3a}{28}$

2. $\frac{x+8}{x^2-25} - \frac{3}{x^2-25}$

3. $\frac{3x}{x-2} - \frac{6}{x-2}$

4. $\frac{3c-1}{c^2-5c+4} - \frac{2c+3}{c^2-5c+4}$

5. $\frac{3x^2}{x^2-1}-\frac{x+4}{x^2-1}$

6. $\frac{2x^2}{x^2-x-12}-\frac{7x+4}{x^2-x-12}$

7. $\frac{6}{x-3}+\frac{2x}{3-x}$

8. $\frac{4a}{3a-5}-\frac{3}{5-3a}$

9. $\frac{5}{25-x^2}-\frac{x}{x^2-25}$

9.3.2 Add and subtract rational expressions that do not have a common denominator.

Steps to add and subtract algebraic fractions:

1. Factor the denominators and find the LCD. Then rewrite each fraction in terms of the LCD.
2. Add (and/or subtract) the numerators. The denominator is the LCD.
3. Simplify - factor the numerator and denominator and divide out any common factors.

Add or subtract, then simplify.

1. $\frac{y}{x} - \frac{4y}{3x} + \frac{3y}{4x}$

2. $\frac{2a}{b} - \frac{3a}{5b} + \frac{7a}{2b}$

3. $\frac{35}{2a-5} + \frac{14a}{5-2a}$

4. $\frac{2t}{3-t} + \frac{6}{t-3}$

5. $x - \frac{5}{7x}$

6. $z - \frac{4}{7-2z}$

7. $\dfrac{6x}{x^2-4}-\dfrac{3}{2-x}$

8. $\dfrac{5t+1}{16-t^2}+\dfrac{7}{t-4}$

9. $\dfrac{x+3}{x^2-2x-8}+\dfrac{3}{4-x}$

10. $\dfrac{2x-1}{x^2-25}+\dfrac{2}{5-x}$

11. $\dfrac{2c}{c-3}-\dfrac{1}{c+1}$

12. $\dfrac{4x}{3x-1}-\dfrac{9}{x+4}$

9.4.1 Complex Fractions

Definition

A **complex fraction** is a fraction whose numerator or denominator contains one or more fractions.

e.g. $\dfrac{1-\frac{1}{x}-\frac{12}{x^2}}{\frac{3}{x}-\frac{4}{x^2}+5}$, $\qquad \dfrac{\frac{3}{2a+1}-5}{4-\frac{7}{2a+1}}$, $\qquad \dfrac{\frac{3}{x-2}-\frac{2}{x+2}}{7+\frac{2x}{x^2-4}}$

To simplify complex fractions

1. Create a simple fraction by multiplying the numerator and denominator of the "main fraction" by the LCD of the "inside fractions."
2. Simplify the remaining fraction so that the only common factor of the numerator and denominator is 1. That is, factor the numerator and denominator of the resulting simple fraction and divide out the common factors.

Simplify each complex fraction.

1. $\dfrac{\frac{1}{a}+\frac{1}{4}}{\frac{1}{a^2}-\frac{1}{16}}$

2. $$\frac{\frac{1}{3}-\frac{1}{t}}{\frac{1}{9}-\frac{1}{t^2}}$$

3. $$\frac{\frac{3}{2a+1}-3}{2-\frac{4a}{2a+1}}$$

4. $$\frac{x-5+\frac{14}{x+4}}{x+3-\frac{2}{x+4}}$$

5. $$\frac{1-\frac{2}{t}-\frac{15}{t^2}}{1-\frac{11}{t}+\frac{30}{t^2}}$$

6. $$\frac{1+\frac{2}{a-2}}{1-\frac{3}{a+2}}$$

9.5.1 Solve equations containing fractions

Complete these steps for problems 1-22, section 9.5.

1. Clear all denominators by multiplying each side of the equation by the LCD of all the fractions in the equation.
2. Solve the resulting equation.
3. Check the solution. Make sure the solution does not result in division by zero when substituted into the original equation.

Solve for x.

1. $\frac{1}{2} - \frac{3}{x} = -\frac{x}{2}$

2. $x - \frac{1}{2} = \frac{3}{x}$

3. $\dfrac{2x}{x-2}=1+\dfrac{4}{x-2}$

4. $\dfrac{-3x}{x+8}=2+\dfrac{4}{x+8}$

5. $\dfrac{8}{x}=1+\dfrac{2}{x-2}$

6. $\dfrac{a}{a+2}+\dfrac{2}{a-2}=\dfrac{a+6}{a^2-4}$

9.5.2 Solve Proportions

Definition

A **proportion** is two equal fractions (as either rates or ratios).

e.g. $\frac{8}{x} = \frac{4}{3}$, $\frac{5}{x+7} = \frac{4}{x}$, $\frac{2}{x+3} = \frac{1}{x+5}$

Complete these steps to solve the proportion problems (exercises 24 to 38, section 9.5)

1. Cross-multiply and solve the resulting equation.
2. Check the solution. Make sure your solution does not result in division by zero when evaluated in the original equation.

Solve for x. (1. - 6; 2. 28; 3. - 7)

1. $\frac{8}{x} = \frac{4}{3}$

2. $\frac{5}{x+7} = \frac{4}{x}$

3. $\frac{2}{x+3} = \frac{1}{x+5}$

9.5.3 Applications of Proportions

Example

1. The monthly loan payment for a car is \$14.60 for each \$500 borrowed. At this rate, find the monthly payment for a \$18000 car loan.

2. Nine tiles will cover 4 square feet of flooring. At this rate, how many square feet can be tiled with 270 tiles.

3. An investment of $1200 earns $96 each year. At the same rate, how much additional money must be invested to earn $128 each year?

4. Three ounces of medication are required for a 120 pound adult. At the same rate, how many additional ounces of the medication are required for a 180 pound adult?

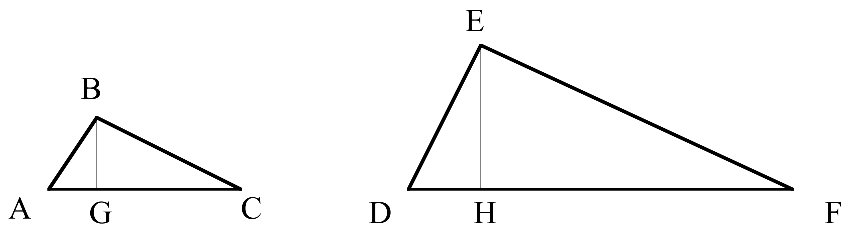
9.5.4 Similar Triangles

Definition

Similar objects have the same shape, but not necessarily the same size.

e.g. A tennis ball is similar to a basketball.
A model house is similar to the actual house.

Fact The ratio of corresponding parts of similar triangles are equal.

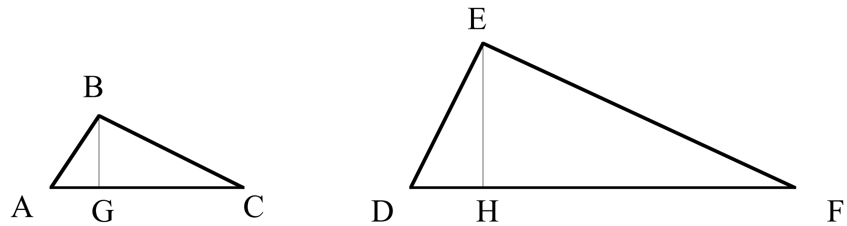


Note $\frac{AB}{AC} = \frac{DE}{\quad}$ $\frac{AB}{BC} = \frac{\quad}{\quad}$ $\frac{AC}{BG} = \frac{\quad}{\quad}$

Example 1

Suppose triangles ABC and DEF are similar with heights BG and EH, respectively.

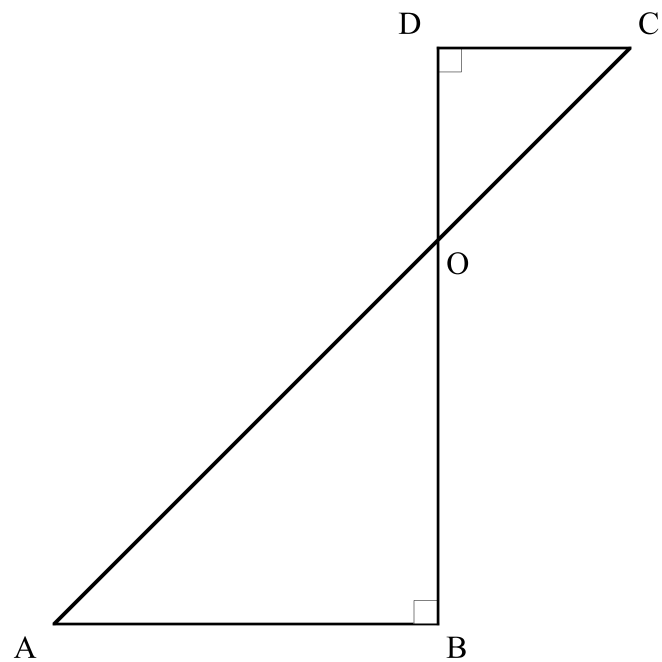
a. If AC = 5 in, DF = 12 in, and EH = 3 in, find the length of BG..



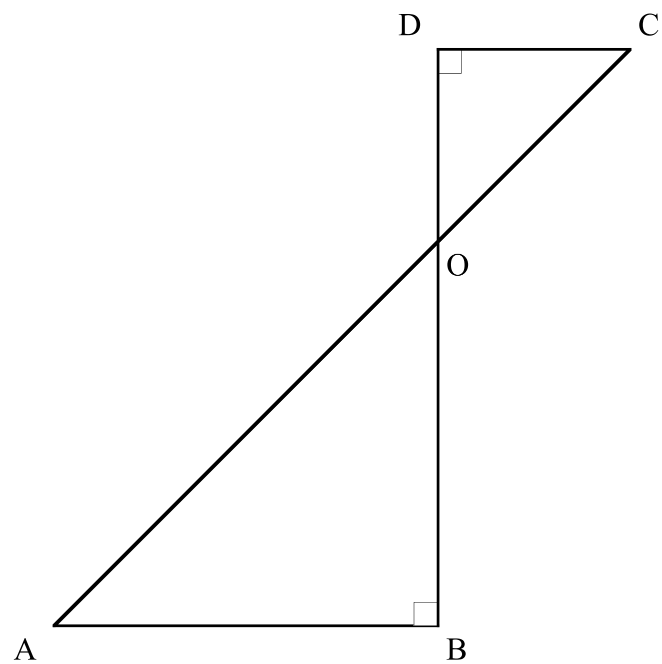
b. If BG = 1.5 in, EH = 3 in, FE = 4 in, and AB = 3 in, find the lengths DE and BC.

Example 2

a. Suppose DC = 4 cm, DO =3 cm, and AB = 7 cm. Find the length of BO.



b. Suppose AB =12 in, DC = 4 in, and AC =18 in. Find the length of CO.



9.6 Literal Equations

Definition

A **literal equation** is an equation that contains more than one variable.

i. $3x - 5y = 15$ ii. $4a + c = 12$

Definition

A **formula** is a literal equation that expresses a relationship among the variables.

i. $d = r \cdot t$ ii. $A = \frac{1}{2}bh$ iii. $R = \frac{C - S}{t}$

Definition

To **solve an equation for a specified variable** means to isolate the variable on one side of the equation; the variable being isolated cannot occur on both sides of the equation.

Example 2

1. Solve for *t*: $A = P + Prt$

2. Solve for *r*: $A = P + Prt$

3. Solve for P: $A = P + Prt$

4. Solve for x: $3x - 4y = 12$

5. Solve for y: $3x - 4y = 12$

6. Solve for R: $I = \dfrac{E}{R + r}$

7. Solve for S: $R = \dfrac{C - S}{t}$

8. Solve for f: $t = fm + gm$

9. Solve for m: $t = fm + gm$

10. Solve for a: $a = aw - w$

11. Solve for w: $a = aw - w$

9.7 - 1 Work Problems

Definition

The **rate of work** is the fraction (or part) of a task completed in one unit of time. If t is the total time to complete a task, then $\frac{1}{t}$ is the rate the task is completed during each unit of time.

Examples

a. If a painter can paint a room in 4 hours, then
   i. the task is painting the room and
   ii. the rate of the painter is 1/4 of the room per hour.

b. If a pipe can fill a tank in 30 in, then
   i. the task is

   ii. the rate is

c. If Doug can roof a house in t hours, then
   i. the task is

   ii. the rate is

Rate or Work Equation

$$Rate \cdot Time = Part\ (or\ fraction)\ of\ Task\ Completed;\ R \cdot T = P$$

where R = the rate the task is completed during each unit of time.
T = the time worked on the task
P = the part (or fraction) of the task completed in time T.

Example 1

Suppose a faucet can fill a sink in 5 minutes.

a. What is the task?

b. What is the rate?

c. What part of the task is completed after 2 minutes? 3 minutes? after 5 minutes?

Fact The whole task being completed is represented by the number 1. That is, the sum of all the parts of the task is 1.

Example 2

A painter can paint a ceiling in 60 min. The painter's apprentice can paint the same ceiling in 90 min. How long will it take to paint the ceiling if they both work together?

a. What is unknown? What is the variable going to represent?

b. Organize the data.

	Rate	·	Time	=	Fraction (Part) Completed
painter		·		=	
apprentice		·		=	

c. Use the fact that the sum of the parts must equal 1 to write and solve the equation. Make sure you answer the specific question asked.

Example 3

A small water pipe takes four times longer to fill a tank than does a large water pipe. With both pipes open, it takes 3 h to fill the tank. Find the time it would take the small pipe, working alone, to fill the tank?

a. What is unknown?

b. Organize the data.

	Rate	·	Time	=	Fraction (Part) Completed
small pipe		·		=	
large pipe		·		=	

c. Use the fact that the sum of the parts must equal 1 to write and solve the equation. Make sure you answer the specific question asked.

Follow-up Questions

d. What is the rate of the small pipe (include units)?

e. What is the rate of the large pipe (include units)?

f. What part of the task was completed by the small pipe?

g. What part of the task was completed by the large pipe?

Example 4

Two computer printers that work at the same rate are working together to print the employee's annual tax forms for a corporation. After working together for 3 h, one of the printers quits. The second printer requires 2 more hours to complete the forms.

a. Find the time it would take one printer, working alone, to print the forms.

	Rate	·	Time	=	Fraction (Part) Completed
computer 1		·		=	
computer 2		·		=	

b. What part of the task did printer 1 complete?

c. What part of the task did printer 2 complete?

d. What was the rate of each printer?

Example 5

It takes Doug 6 days to roof a house. If Doug's son helps him, the job can be completed in 4 days. How long would it take Doug's son, working alone, to roof a house. (ans: 12 hours)

	Rate	·	Time	=	Fraction (Part) Completed
Doug		·		=	
Doug's son		·		=	

Example 6

A large heating unit and a smaller heating unit are being used to heat a pool. The large unit, working alone, requires 8 h to heat the pool. After both units have been operating for two hours, the large units breaks down and quits. The small unit requires 9 more hours to heat the pool. How long would it take the small unit, working alone, to heat the pool? (ans: $14\frac{2}{3}$ hours)

	Rate	·	Time	=	Fraction (Part) Completed
small heater		·		=	
large heater		·		=	

9.7-2 Uniform Motion Problems

Definition

Uniform motion is motion at a constant speed. The **uniform motion equation** takes the following two forms.

1. $Rate \cdot Time = Distance,\ R \cdot T = D$

This equation (from section 4.7) is useful when the relationship of distances traveled is known. For example, In "round trip" problems you know the distance in one direction is the same as the return distance; the equation is generated by setting "Distance There" = "Distance Back."

2. $Distance \div Rate = Time,\ D \div R = T$

This form of the uniform motion equation is useful when the relationship of the travel times is known. Such as, the total time is known, or the two times given are equal.

Example 1

The speed of a boat in still water is 20 mph. The boat traveled 120 mi down a river in the same amount of time it took to boat to travel 80 mi up the river. Find the rate of the river's current. (ans: 4 mph)

	Distance	÷	Rate	=	Time
down river		÷		=	
trip back		÷		=	

Example 2

A cyclist rode the first 20 mi of a trip at a constant rate. For the next 16 mi, the cyclist reduced her speed by 2 mph. The total time for the trip was 4 h. Find the rate of the cyclist for each part of the trip. (ans: 10 mph, 8 mph)

a. Which form of the uniform motion equation is most What is the unknown?

b. Organize the data. Solve the problem.

	Distance	÷	Rate	=	Time
part 1		÷		=	
part 2		÷		=	

Example 3

The total time for a sailboat to sail back and forth across a lake 6 km wide was 3 h. The rate sailing back was twice the rate sailing across the lake. Find the rate of the sailboat going across the lake. (ans: 2. mph)

Example 4

A car traveling at a rate of 60 mph overtakes a bus traveling at 45 mph. If the bus had a 1-hr head start, how far from the starting point does the car overtake the bus? (ans: 180 miles)

	Distance	÷	Rate	=	Time
car		÷		=	
bus		÷		=	

Example 5

At 10 a.m. a plane leaves Boston for Seattle, a distance of 3000 mi. One hour later a plane leaves Seattle for Boston. Both planes are traveling at 500 mph. How many hours after the plane leaves Seattle will the planes pass each other? (ans: 3.5 hours)

	Rate	·	Time	=	Distance
Boston-Seattle		·		=	
Seattle-Boston		·		=	