

9.1 Simplify Radical Expressions

Radical Notation for the n -th Root of a

If n is an integer greater than one, then the **n th root of a** is the number whose n th power is a . There are two notations for the n th root of a :

$$\sqrt[n]{a} = a^{1/n}$$

where n is called the **index of the radical**

$\sqrt{\quad}$ is called the **radical symbol**

a is called the **radicand**

$\sqrt[n]{a}$ is the **radical form** of the n -th root of a

$a^{1/n}$ is the **exponential form** of the n -th root of a

An expression containing a radical symbol is called a **radical expression**. Some examples of radical expressions are

$$\sqrt[3]{8-x}, \quad 3\sqrt{a} - 4\sqrt[5]{b}, \quad \frac{3 - \sqrt[3]{4}}{\sqrt[3]{5}}$$

Consider the Sign of the Radicand a : Positive, Negative, or Zero

1. If a is **positive**, then the n th root of a is also a *positive* number - specifically the positive number whose n th power is a .

e.g. $\sqrt[3]{125}$ is asking $(\quad)^3 = 125$

$\sqrt[4]{16}$ is asking $(\quad)^4 = 16$

2. If a is **negative**, then n must be odd for the n th root of a to be a real number.

e.g. $\sqrt[3]{-125}$ is asking $(\quad)^3 = -125$

$\sqrt[4]{-16}$ is asking $(\quad)^4 = -16$

Furthermore, if a is negative and n is odd, then the n th root of a is also a *negative* number - specifically the negative number whose n th power is a .

3. If a is **zero**, then $\sqrt[n]{0} = 0$.

Example 1

1. Evaluate $\sqrt[4]{81}$

2. Evaluate $\sqrt[3]{-64}$

3. Evaluate $\sqrt[4]{-27}$

4. Evaluate $\sqrt[2]{0}$

Square Roots and Cube Roots

1. The second root of a is called the **square root of a** .

i.e. $\sqrt{a} = \sqrt[2]{a}$ is read “the square root of a ”

2. The third root of a is called the **cube root of a** .

i.e. $\sqrt[3]{a}$ is read “the cube root of a ”

Definition of $a^{\frac{m}{n}}$

If $a^{1/n} = \sqrt[n]{a}$ is a real number, then

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

where $a^{\frac{m}{n}}$ is the **exponential form** of the expression, and $\sqrt[n]{a^m}$ is the **radical form** of the expression.

Example 2

Put each expression in radical form.

1. $y^{\frac{3}{7}}$

2. $(4c - d)^{\frac{2}{5}}$

3. $(7 - 3a)^{\frac{9}{14}}$

Example 3

Put each expression in exponential form.

1. $\sqrt[4]{c^3}$

2. $\sqrt[3]{(2x + 5)^2}$

3. $(\sqrt[4]{9 - 4b})^5$

Example 4

Simplify each expression (reduce the index).

1. $\sqrt[15]{x^5}$

2. $\sqrt[3]{(2x+5)^3}$

3. $(\sqrt[4]{a-4b})^8$

Product Property of RadicalsIf $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$$

In words this tells us the n th root of the product is the product of n th roots. In terms of the order of operations, when the only operations are n th-rooting and multiplying, then it does not matter which operation comes first.

Proof

$$\begin{aligned}\sqrt[n]{ab} &= (ab)^{\frac{1}{n}} \\ &= a^{\frac{1}{n}}b^{\frac{1}{n}} \\ &= \sqrt[n]{a}\sqrt[n]{b}\end{aligned}$$

Condition #1 for a Simplified Radical Expression

A radical expression $\sqrt[n]{a}$ is *not* simplified when the radicand a has any perfect n th-power factors.

Example 1

Since $n = 2$, the radicand a can have no perfect square factors.

Simplify $\sqrt{8}$

Simplify $\sqrt{28y^{17}}$

Example 2

Since $n = 3$, the radicand a can have no perfect cube factors.

Simplify $\sqrt[3]{16}$

Simplify $\sqrt[3]{54x^7}$

Example 3

Since $n = 4$, the radicand a can have no 4th-power factors.

Simplify $\sqrt[4]{32x^7}$

Simplify $\sqrt[4]{162(3x + 5)^9}$

Example 4 Simplify each expression.

1. $\sqrt{75x^6}$

2. $\sqrt{(2x + 7)^9}$

Example 5 Simplify each expression.

1. $\sqrt[3]{40x^{13}}$

2. $\sqrt[4]{80x^{24}y^{15}}$

3. $\sqrt[5]{(3x+11)^{14}}$

4. $\sqrt{45a^{11}b^4}$

Condition #2 for a Simplified Radical Expression

The radical expression $\sqrt[n]{a^m}$ is *not* simplified if m and n have any common factors. That is, m/n must be in simplest terms.

Example 6

Simplify each expression.

1. $\sqrt[6]{x^3}$

2. $\sqrt[15]{y^3}$

3. $\sqrt[4]{64x^6}$

9.2 Add, Subtract, and Multiply Radical Expressions

Consider the radical expression $3\sqrt[5]{b} + 7\sqrt[5]{b}$

Use the distributive property to rewrite the expression in factored form. Then simplify the expression.

Like Terms & Combining Like Terms

Like terms have identical variable and radical factors. To **combine like terms** means to add the coefficients while leaving the variable and radical factors unchanged.

To add or subtract radical terms means to combine like terms

Example 1

Perform the indicated operation.

1. $3\sqrt{x} + 6\sqrt{x}$

2. $5\sqrt[5]{3x} - 2\sqrt[5]{3x}$

3. $4\sqrt[3]{x} + 5\sqrt[6]{x}$

4. $3\sqrt[4]{x} - 2\sqrt[4]{x+1}$

Example 2

Perform the indicated operation.

1. $3\sqrt[4]{x} + 4\sqrt{x} + 2(\sqrt[4]{x} + 7\sqrt{x})$

2. $3(\sqrt[3]{x+1} - 2) - 4\sqrt[3]{x+1}$

Example 3

Perform the indicated operation.

1. $\sqrt{8x} + 3\sqrt{2x}$

2. $-2\sqrt[3]{16x^4} + 5x\sqrt[3]{54x}$

Multiply Radical Expressions

Product Property of Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$.

Specifically, if $n = 2$ and $a = b$, then $\sqrt{a}\sqrt{a} = \sqrt{a^2} = a$

Example 2

Perform the indicated operation.

1. $\sqrt{2x}\sqrt{2x}$

2. $\sqrt{2y-5}\sqrt{2y-5}$

Example 3

Perform the indicated operation and simplify.

1. $2\sqrt{6x}(-4\sqrt{3x})$

2. $3\sqrt{7x}(4\sqrt{x} - \sqrt{7})$

Example 4

Perform the indicated operation and simplify.

1. $(2\sqrt{y} - 7)(3\sqrt{y} + 4)$

2. $(3\sqrt{x} + 5)^2$

Example 4

Simplify $(-2\sqrt{x})\sqrt[4]{x}$

- i. Write the expression in exponential form
- ii. Perform the indicated operation(s)
- iii. Write the expression in radical form

9.3 Rationalizing Denominators and Simplifying Quotients of Radical Expressions

A simplified radical expression cannot have a radical in the denominator. The procedure for removing a radical from the denominator is called **rationalizing the denominator**. The product property of radicals is used to rationalize a denominator.

Product Property of Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$.

Specifically, if $n = 2$ and $a = b$, then $\sqrt{a}\sqrt{a} = \sqrt{a^2} = a$

Example 1 Rationalize a One-Term, Square Root ($n = 2$) Denominator

1. Simplify (rationalize the denominator) $\frac{2}{\sqrt{7}}$

2. Simplify (rationalize the denominator) $\frac{3}{\sqrt{15x}}$

Example 2 Rationalize a One-Term, Cube Root ($n = 3$) Denominator

Note
 $\sqrt[3]{a^3} = a$

1. Simplify (rationalize the denominator) $\frac{5}{\sqrt[3]{9}}$

$$\frac{5}{\sqrt[3]{9}} = \frac{5}{\sqrt[3]{3^2}} = \frac{5}{\sqrt[3]{3^2}} \cdot \frac{\sqrt[3]{3}}{\sqrt[3]{3}} = \frac{5\sqrt[3]{3}}{\sqrt[3]{3^3}} = \frac{5\sqrt[3]{3}}{3}$$

the goal is to make the radicand a perfect cube

2. Simplify (rationalize the denominator) $\frac{3}{\sqrt[3]{5x^2}}$

3. Simplify (rationalize the denominator) $\frac{7}{\sqrt[3]{4x}}$

Properties of Radicals

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

Product Property

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$$

Quotient Property

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Simplified Radical Expression

A radical expression is simplified if

1. There are no radicals in a denominator.
2. There are no fractions inside a radical symbol.
3. All radicands have no n th power factors.
4. The numerator and denominator of any rational expression (fractions) have no common factors.

Example 3

1. Simplify $\sqrt{\frac{7}{y}}$

2. Simplify $\sqrt[3]{\frac{11}{3x}}$

3. Simplify $\sqrt[4]{\frac{7}{4d^3}}$

Rationalize a Two-Term Denominator

Conjugate

The **conjugate** of the two-term expression $a + b$ is $a - b$ and visa versa.

Example 4

For each of the following, identify the conjugate of the expression. Then find the product of the expression and its conjugate.

Expression	Conjugate	Product
$a + b$	$a - b$	
$a - \sqrt{3}$		
$\sqrt{x} - 7$		
$2\sqrt{3} + 4\sqrt{5}$		

Fact The product of a square-root expression and its conjugate is an expression containing no square roots (i.e. a rational expression).

Example 5

Simplify $\frac{7}{\sqrt{x+4}}$

Example 6

Simplify $\frac{\sqrt{a}+7}{3\sqrt{a}-2}$

9.5 Solve Square Root Equations

Recall that *expressions* are things we can be asked to simplify, add, subtract, multiply, and divide. However, *equations* (two equal expressions) are things we are asked to solve. In this section we will solve square root equations, such as,

$$\sqrt{x} = 7 \qquad 2\sqrt{x} - 5 = 11 \qquad 3 = \sqrt{6+x} + \sqrt{x}$$

To Solve an Equation Containing One Square Root Term

1. Isolate the square root term on one side of the equation.
2. Square both sides of the equation and solve.
3. Check the solution(s) in the original equation.

Example 1

1. Solve $\sqrt{x} - 7 = 0$

2. Solve $2\sqrt{x} - 5 = 11$

3. Solve $3\sqrt{x} + 20 = 2$

Watch for Extraneous Solutions

When both sides of an equation are squared it is possible for the modified equation to have a solution that does not satisfy the given equation - these false solutions are called **extraneous solutions** and must be discarded.

Example 2

1. Solve $2\sqrt{x} - 12 = -7\sqrt{x} + 16$

2. Solve $\sqrt{4-x} - x + 5 = 7$

3. Solve $\sqrt{13+4x} = x + 4$

Example 3 Solve each equation.

1. $\sqrt{x+1} - \sqrt{x-2} = 1$

2. $\sqrt{3x+4} - \sqrt{2x-4} = 2$

3. $\sqrt{\sqrt{x}+1} = 2$

Example 4 Solve $\frac{1}{\sqrt{x+2}} = 3 - \sqrt{x+2}$

Example 5

1. Find the zeros & x -intercepts of $f(x) = \sqrt{3x+4} - \sqrt{2+5x}$

2. Find the y -intercept of f .

3. Verify the results by graphing f on your calculator.

